Math Camp for Economists:

Matrix Analysis

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Summer 2023

Land acknowledgement

I acknowledge and respect the Lekwungen peoples on whose traditional territory the University of Victoria stands, and the Songhees, Esquimalt, and WSÁNEĆ peoples whose historical relationships with the land continue to this day

Overview of this lesson

In this lesson we're going to non-rigorously cover the basics of linear algebra:

- Basic properties and operations with matrices
- Special kinds and forms of matrices
- Determinants, adjoints, and inverses of matrices
- Quadratic forms, principal minors, and definiteness of matrices

What is a matrix?

A matrix, A, of dimension $k \times n$ is just a rectangular array of numbers with k rows and n columns

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{k1} & a_{k2} & \cdots & a_{kj} & \cdots & a_{kn} \end{pmatrix}$$

The number in row i and column j is the (i, j)th entry, written a_{ij}

Some basic properties of matrices

- Two matrices, A and B, have the same "size" if they have the same dimensions, $k \times n$
- Two matrices, A and B, are equal if they are same-sized and each corresponding element is equal
- If the sizes are right, two matrices can be added, subtracted, multiplied, and divided
- Interchanging the rows and columns of a $k \times n$ matrix, A, yields a $n \times k$ matrix, A^T (or A'), called the "transpose" of A

$$A = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \Rightarrow A' = \begin{pmatrix} a & d \\ b & e \\ c & f \end{pmatrix}$$

- A $k \times 1$ matrix is a "column vector". A $1 \times n$ matrix is a "row vector"

$$A = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \Rightarrow A' = \begin{pmatrix} a & b & c \end{pmatrix}$$

Maxtrix transposition exercises

Exercise:

(1) Create a 3×3 matrix, W, using real numbers

(2) Create a different 3×3 matrix, Z, using real numbers

(3) Create a 2×3 matrix, V, using real numbers

(4) Find the transpose of W

(5) Find the transpose of Z

(6) Find the transpose of V

A bit more on vectors

Vectors can be thought of as points in n-space. They can also be thought of as "displacements"—that is, they have magnitude and direction (often written with commas instead of spaces between elements)

- In $\mathbb{R}^2$, a vector (x, y) is an ordered "pair" where x and y are real numbers
  - → The displacement $(3, 2)$ means move 3 units right (along the x-axis) and 2 units up (along the y-axis) relative to the current location
- In $\mathbb{R}^3$, a vector (x, y, z) is an ordered "triple" where x and y and z are real numbers
  - → The displacement $(1, -4, 1)$ means move 1 unit right (along the x-axis), 4 units down (along the y-axis), and 1 unit forward (along the z axis) relative to the current location
- More generally, in $\mathbb{R}^n$, a vector $(x_1, ..., x_n)$ is a "tuple" (a finite sequence) of real numbers
- The tail of the arrow is not specified by the vector, and can be anywhere in Euclidean space
- The natural initial location is the origin, $\mathbf{0}$

Adding matrices

Two matrices, A and B, can be added only if they have the same size

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

$$\Rightarrow A + B = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} & a_{13} + b_{13} \\ a_{21} + b_{21} & a_{22} + b_{22} & a_{23} + b_{23} \\ a_{31} + b_{31} & a_{32} + b_{32} & a_{33} + b_{33} \end{pmatrix}$$

But if we have

$$C = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{pmatrix}, \quad D = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \\ d_{31} & d_{32} \end{pmatrix}$$

... then $C + D$ is not defined

Exercise:

(1) Is $C + D'$ defined? If so, what is it? What about $C' + D$? And $C' + D'$?

Subtracting matrices

Two matrices, A and B, can be subtracted only if they have the same size

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

$$\Rightarrow A - B = \begin{pmatrix} a_{11} - b_{11} & a_{12} - b_{12} & a_{13} - b_{13} \\ a_{21} - b_{21} & a_{22} - b_{22} & a_{23} - b_{23} \\ a_{31} - b_{31} & a_{32} - b_{32} & a_{33} - b_{33} \end{pmatrix}$$

But if we have

$$C = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{pmatrix}, \quad D = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \\ d_{31} & d_{32} \end{pmatrix}$$

... then $C - D$ is not defined

Exercise:

(1) Is $C - D'$ defined? If so, what is it? What about $C' - D$? And $C' - D'$?

More exercises adding and substracting matrices

Consider your matrices, W, Z, and V

Exercise:

(1) Are $W + Z$, $W' + Z$, $W + Z'$, or $W' + Z'$ defined? If so, what are they?

(2) What about $W + V$, $W' + V$, $W + V'$, or $W' + V'$?

(3) Are $W - Z$, $W' - Z$, $W - Z'$, or $W' - Z'$ defined? If so, what are they?

(4) What about $W - V$, $W' - V$, $W - V'$, or $W' - V'$?

Scalar multiplication of matrices

Matrices can be multiplied by ordinary numbers, called "scalars". The product of a scalar r and a matrix A is rA

$$rA = r\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} ra_{11} & ra_{12} & ra_{13} \\ ra_{21} & ra_{22} & ra_{23} \\ ra_{31} & ra_{32} & ra_{33} \end{pmatrix}$$

Multiplying matrices

Two matrices, A and B, can be multiplied iff the number of columns of A = the number of rows of B

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} \\ b_{21} \\ b_{31} \end{pmatrix} \Rightarrow AB = a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31}$$

In general, if A and B are matrices than can be multiplied, then the (i, j)th entry of the (dot) product, AB, is the product of the ith row of A and the jth row of B: $\sum_{h=1}^{3} a_{ih}b_{hj}$

$$\begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} aA + bC & aB + bD \\ cA + dC & cB + dD \\ eA + fC & eB + fD \end{pmatrix}$$

But note this is not defined

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix}$$

Matrix multiplication exercises

Consider your matrices, W, Z, and V

Exercise:

(1) Is WZ defined? If so, what is it? What about ZW? And $W'Z$? And $W'Z'$?

(2) Is WV defined? If so, what is it? What about $W'V$? And WV'? And $W'V'$?

(3) Is VV' defined? If so, what is it? What about $V'V$? And WVV'? And $WV'V$?

A bit more on multiplying matrices

- If A is $k \times m$ and B is $m \times n$, then the product AB will be $k \times n$, inheriting the number of rows from A and the number of columns from B
- The "identity" matrix is $n \times n$ with $a_{ii} = 1$ for all i and $a_{ij} = 0$ for all $i \neq j$

$$I = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

- I has several useful properties:
  (1) For any $m \times n$ matrix A, $AI = A$
  (2) For any $n \times l$ matrix B, $IB = B$
  (3) $II = I$

Useful laws of matrix algebra

Let $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ be $k \times n$ matrices, with $k > 1$, $n > 1$. Let $\mathbf{a}$ and $\mathbf{b}$ be $k \times 1$ vectors, and $\mathbf{c}$ be a $n \times 1$ vector

- $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$
- $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$
- $(\mathbf{A} + \mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC}$
- $(\mathbf{AB})' = \mathbf{B}'\mathbf{A}'$
- $(\mathbf{a}'\mathbf{B}\mathbf{c})' = \mathbf{c}'\mathbf{B}'\mathbf{a}$
- $\mathbf{a}'\mathbf{b} = \mathbf{b}'\mathbf{a}$
- In general, $\mathbf{AB} \neq \mathbf{BA}$

More matrix exercises

We've just seen several useful laws of matrix algebra:

Exercise:

(1) Find the dimensionality of each LHS and RHS of these laws given the defined matrices

(2) For each law, come up with a concrete example (with $k > 2$ and $n > 1$)

(3) Use some of these results to show that $\mathbf{a}'\mathbf{B}\mathbf{c} = (\mathbf{a}'\mathbf{B}\mathbf{c})'$. Explain why this is true, and come up with an example to demonstrate it

Special kinds of matrices

Here are several important classes of $k \times n$ matrices

- Square matrix: $k = n \Rightarrow$ A square matrix is a $n \times n$ matrix. Also identified as a matrix "in $\mathbf{M}_n$"
  - $\rightarrow$ If a matrix $\mathbf{X}$ is $m \times n$, $m \neq n$, then $\mathbf{X}$ is not square. But $\mathbf{X}'\mathbf{X}$ is square
- Diagonal matrix: A $n \times n$ matrix for which and $a_{ij} = 0$ for $i \neq j$
- Upper-triangular matrix: A matrix, $\mathbf{A}$, whose elements $a_{ij} = 0$ if $i > j$
- Lower-triangular matrix: A matrix, $\mathbf{A}$, whose elements $a_{ij} = 0$ if $i < j$
- Symmetric matrix: A matrix, $\mathbf{A}$, such that $\mathbf{A}' = \mathbf{A} \iff a_{ij} = a_{ji}$ for all i, j. Necessarily $n \times n$
- Idempotent matrix: A $n \times n$ matrix $\mathbf{A}$ for which $\mathbf{AA} = \mathbf{A}$ (e.g. $\mathbf{A} = \mathbf{I}$)
- Permutation matrix: $n \times n$ and only 0s and 1s, with exactly one 1 in each row and column
- Nonsingular matrix: A $n \times n$ matrix $\mathbf{A}$ whose rank is n: $rank(A) = n$
  - $\rightarrow$ $rank(A) =$ the number of linearly independent columns (or rows) of A
  - $\rightarrow$ iff $\mathbf{Ax} = \mathbf{b}$ has a unique solution $\mathbf{x}$ for every RHS b

Exercise: Come up with an example of each of these special types of matrices where $k > 1$ and $n > 2$

Reduced row echelon form

- A $k \times n$ matrix $\mathbf{A}$ is in "row echelon form" if each row has more leading zeros than the row preceding it
- The first nonzero entry in each row of a $k \times n$ matrix $\mathbf{A}$ is called a "pivot"
- A row echelon $k \times n$ matrix $\mathbf{A}$ in which each pivot is a 1 and in which each column containing a pivot contains no other non-zero entries is in "reduced row echelon form" (rref)

Exercise:

(1) Write a non-zero 2×2 matrix that is not in row echelon form

(2) Write a non-zero 2×2 matrix that is in row echelon form

(3) What are the pivots of your matrix from (2)?

(4) Write a 2×2 matrix in reduced row echelon form (rref)

(5) Write a 3×3 matrix in rref that is not the identity matrix and which has at least two pivots

Augmented matrices

Let **A** be a $k \times n$ matrix and **B** be a $k \times m$ matrix (same number of rows)

- Then we can "augment" **A** with **B** by writing

$$\left(\begin{array}{ccc:ccc} a_{11} & \dots & a_{1n} & b_{11} & \dots & b_{1m} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{k1} & \dots & a_{kn} & b_{k1} & \dots & b_{km} \end{array}\right)$$

  which we denote $[\mathbf{A}|\mathbf{B}]$ and call an "augmented matrix"

  $\rightarrow$ Do *not* think of this as conditionality (or "A given B")

Augmented matrices and elementary row operations

Suppose we have a system of 2 equations with 2 unknowns:

$$3x + y = 1$$
$$x - y = 2$$

We can solve this simple system using e.g. elimination to find $(x^*, y^*) = (0.75, -1.25)$

We can also put it in matrix form, $\mathbf{Ax} = \mathbf{b}$, and solve for $\mathbf{x}$ by augmenting $\mathbf{A}$ with $\mathbf{b}$ and using "elementary row operations" on the result to reduce the $\mathbf{A}$ part to reduced row echelon form

Exercise:

(1) Use elimination to show that $(x^*, y^*) = (0.75, -1.25)$

(2) Write down $\mathbf{A}$, $\mathbf{x}$, and $\mathbf{b}$, then write out the system in matrix form using these matrices

(3) Write the augmented matrix of this system, $[\mathbf{A}|\mathbf{b}]$

Performing elementary row operations on an augmented matrix

"Elementary row operations" involve one or more iterations of one or more of the following operations:

(1) Interchange two rows of a matrix

(2) Change a row by adding it to a multiple of another row

(3) Multiply each element in a row by the same nonzero number

To solve a system of equations, we perform elementary row operations on each row in the augmented matrix until the $\mathbf{A}$ component is in reduced row echelon form

$\rightarrow$ Do this iteratively to ensure you don't make a mistake

$\rightarrow$ The right-most column in the final augmented matrix will be the solution, $\mathbf{x}^*$

Exercise:

(1) Perform elementary row operations on $[\mathbf{A}|\mathbf{b}]$ from the previous exercise to solve for $\mathbf{x}$

(2) Multiply $\mathbf{A}\mathbf{x}^*$. What is the result?

(3) Check if your matrix $\mathbf{W}$ (from earlier) is nonsingular. If not, change it so it is. Now augment this nonsingular matrix $\mathbf{W}$ with $\mathbf{I}$, then reduce the $\mathbf{W}$ component to rref

Determinants of matrices

The "determinant" of a matrix $\mathbf{A}$, written as $\det(\mathbf{A})$ or $|\mathbf{A}|$, is defined inductively

Definitions of the determinant for 1×1 and 2×2 matrices:

- For a 1×1 matrix, the scalar a, the inverse $1/a$ exists iff $a \neq 0$
  - $\det(a) = a$
- For a 2×2 matrix, $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\mathbf{A}$ is nonsingular iff $ad - bc \neq 0$
  - $$\det(\mathbf{A}) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21}$$
    $$= a_{11}\det(a_{22}) - a_{12}\det(a_{21})$$

  → The first term on the RHS is a_{11} times the determinant of the submatrix obtained by deleting row 1 and column 1 from $\mathbf{A}$

  → The second term on the RHS is a_{12} times the determinant of the submatrix obtained by deleting row 1 and column 2 from $\mathbf{A}$

  → The terms alternate in sign: the first receives a positive, the second a negative

Determinants of matrices (general)

Let $\mathbf{A}$ be a $n \times n$ matrix with $n > 1$. Let $\mathbf{A}_{ij}$ be the $(n-1) \times (n-1)$ submatrix obtained by deleting row i and column j from $\mathbf{A}$. Then

- The scalar $M_{ij} \equiv \det(\mathbf{A}_{ij})$ is called the (i,j)th "minor" of $\mathbf{A}$
- The scalar $C_{ij} \equiv (-1)^{i+j} M_{ij}$ is called the (i,j)th "cofactor" of $\mathbf{A}$

  $\rightarrow$ $M_{ij} = C_{ij}$ is signed positive if $(i+j)$ is even, and $M_{ij} = -C_{ij}$ is negative if $(i+j)$ is odd

Then the determinant of an $n \times n$ matrix $\mathbf{A}$ is

$$\begin{aligned} \det(\mathbf{A}) &= a_{11}C_{11} + a_{12}C_{12} + \ldots + a_{1n}C_{1n} \\ &= a_{11}M_{11} - a_{12}M_{12} + \ldots + (-1)^{n+1}a_{1n}M_{1n} \end{aligned}$$

Exercise:

(1) Write down a 3×3 nonsingular matrix. Call it $\mathbf{C}$

(2) Find a matrix $D = aIC$, where $a = 3$ is a scalar and I is the 3×3 identity matrix

(3) Use the method above to find the determinant of D

The determinants you'll need to know from a practical point of view

Let $a \neq 0$ be a scalar, and let $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ be nonsingular matrices

- $|a| = a$
- $|\mathbf{A}| = ad - bc$
- $|\mathbf{B}| = aei + bfg + cdh - ceg - afh - bdi$

Adjoint of a matrix

Let **A** be a $n \times n$ matrix. Let $\mathbf{A}_{ij}$ be the $(n-1) \times (n-1)$ submatrix obtained by deleting row i and column j from **A**. We already saw:

- The scalar $M_{ij} \equiv \det(A_{ij})$ is called the (i,j)th "minor" of **A**
- The scalar $C_{ij} \equiv (-1)^{i+j} M_{ij}$ is called the (i,j)th "cofactor" of **A**

Beyond helping us find determinants, this is useful because

- The "cofactor matrix" of a matrix **A** is the matrix of the cofactors of **A**
- The "adjoint" of a matrix, **A**, denoted $adj(\mathbf{A})$, is the transpose of the cofactor matrix of **A**
  - → Also called the "adjugate" of **A**

Exercise: Let $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ be nonsingular matrices

(1) Find $adj(\mathbf{A})$

(2) Find $adj(\mathbf{B})$

Invertible matrices

- Let $\mathbf{A}$ be a matrix in $\mathbf{M}_n$. The matrix $\mathbf{B}$ in $\mathbf{M}_n$ is an "inverse" for $\mathbf{A}$ if $\mathbf{AB} = \mathbf{BA} = I$
  - → A matrix $\mathbf{A}$ in $\mathbf{M}_n$ can have at most one inverse
- If a matrix $\mathbf{A}$ in $\mathbf{M}_n$, then we denote its unique inverse as $\mathbf{A}^{-1}$
  - → If $\mathbf{A}$ is a 1×1 matrix, $\mathbf{A}^{-1} = \frac{1}{\mathbf{A}}$, so multiplying by $\mathbf{A}^{-1}$ is the analogue of dividing by $\mathbf{A}$
- A matrix $\mathbf{A}$ in $\mathbf{M}_n$ is "invertible" iff it is nonsingular
- If a matrix $\mathbf{A}$ in $\mathbf{M}_n$ is "invertible", then it is nonsingular and the unique solution to the system of linear equations $\mathbf{Ax} = \mathbf{b}$ is $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$
  - → **Exercise:** Prove this given what we've learned
- An $n \times n$ matrix $\mathbf{A}$ is nonsingular (thus invertible) iff $|\mathbf{A}| \neq 0$
  - → Then $\mathbf{A}^{-1} = \dfrac{1}{|\mathbf{A}|} \times adj(\mathbf{A})$
    - You can also find $\mathbf{A}^{-1}$ by creating the augmented matrix $[A|I]$, then using elementary row operations to turn it into $[I|\mathbf{A}^{-1}]$

if we have a system of linear equations, $\mathbf{Ax} = \mathbf{b}$

Useful rules with invertible, square matrices

Let $\mathbf{A}$ and $\mathbf{B}$ be invertible, square matrices. Then:

(1) $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$

(2) $(\mathbf{A}')^{-1} = (\mathbf{A}^{-1})'$

(3) $\mathbf{AB}$ is invertible, and $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$

Exercise: For each of these rules, come up with a concrete example when $n > 1$

Lab exercises 1

Exercise: Suppose you have 4 matrices:

$$\mathbf{W} = \begin{pmatrix} 2 & 3 \\ 9 & 7 \end{pmatrix}, \mathbf{X} = \begin{pmatrix} 6 & 0 & 2 \\ 3 & 4 & 1 \\ 3 & 7 & 7 \end{pmatrix}, \mathbf{Z} = \begin{pmatrix} 6 & 2 & 4 \\ 3 & 4 & 1 \\ 3 & 1 & 2 \end{pmatrix}, \text{ and } \mathbf{V} = \begin{pmatrix} 2 & 1 \\ 2 & 3 \\ 3 & 1 \end{pmatrix}$$

(1) Find $\mathbf{W}^{-1}$ if it exists

(2) Find $\mathbf{X}^{-1}$ if it exists

(3) Find $\mathbf{Z}^{-1}$ if it exists

(4) Find $\mathbf{V}^{-1}$ if it exists

(5) Find $(\mathbf{V}\mathbf{V}')^{-1}$ if it exists

(6) Find $(\mathbf{V}'\mathbf{V})^{-1}$ if it exists

Lab exercise 2

Suppose you have the system of linear equations we looked at earlier:

$$\begin{aligned} 3x + y &= 1 \\ x - y &= 2 \end{aligned}$$

We said this can be put in matrix form, $\mathbf{Ax} = \mathbf{b}$, and since $\mathbf{A}^{-1}$ exists we can solve for $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$

Exercise:

(1) Show that $\mathbf{A}^{-1}$ exists

(2) Prove that the existence of $\mathbf{A}^{-1}$ means that the unique solution to $\mathbf{Ax} = \mathbf{b}$ is $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$. Show each step

(3) Find $\mathbf{A}^{-1}$

(4) Solve for $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$

Lab exercise 3

Exercise: Let $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\mathbf{X} = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

(1) Find $\mathbf{A}^{-1}$ using the method $\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|} \times adj(\mathbf{A})$

(2) Find $\mathbf{A}^{-1}$ by creating the augmentented matrix $[\mathbf{A}|\mathbf{I}]$, then use elementary row operations to reduce the $\mathbf{A}$ component to rref. What is the resulting $\mathbf{I}$ portion?

(3) Find $\mathbf{A}^{-1}\mathbf{A}$

(4) Is $\mathbf{X}$ invertible? How do you know? What about $\mathbf{X}\mathbf{X}'$? What about $\mathbf{X}'\mathbf{X}$?

(5) Find $(\mathbf{X}'\mathbf{X})^{-1}$ Why is it necessary that $\mathbf{X}'\mathbf{X}$ be nonsingular for this?

(6) Find $(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{X}.Showyourwork$

Quadratic forms

We are often interested in “quadratic forms”. A quadratic form on $\mathbb{R}^k$ is a real-valued function of the form

$$Q(x_1, ..., x_k) = \sum_{i,j=1}^{k} a_{ij} x_i x_j$$

For example, the quadratic form in $\mathbb{R}^2$ is $Q(x_1, x_2) = a_{11}x_1^2 + a_{12}x_1x_2 + a_{22}x_2^2$

A quadratic form can also be represented in matrix form given $\mathbf{x}' = [x_1 \ldots x_n]$ and some $n \times n$ matrix $\mathbf{A}$ such that $Q(x_1, ..., x_k) = \sum_{i \le j} a_{ij} x_i x_j = \mathbf{x}'\mathbf{A}\mathbf{x}$

There are many matrices that would do this. The ideal one is symmetric, with the coefficients a_{ij} for $i \neq j$ equally apportioned (halved) between the associated non-diagonal elements of the matrix

The general quadratic form

The "general quadratic form" on R^n, $Q(x_1, ..., x_n) = \sum_{i \leq j} a_{ij} x_i x_j$, can be written as

$$Q(\mathbf{x}) = \begin{pmatrix} x_1 & x_2 & \dots & x_n \end{pmatrix} \begin{pmatrix} a_{11} & \frac{1}{2}a_{12} & \dots & \frac{1}{2}a_{1n} \\ \frac{1}{2}a_{12} & a_{22} & \dots & \frac{1}{2}a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2}a_{1n} & \frac{1}{2}a_{2n} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$= \mathbf{x}'\mathbf{A}\mathbf{x}$$

where **A** is a unique symmetric matrix that defines this quadratic form

Principal minors of a matrix

Let **A** be a $n \times n$ matrix, with $k \leq n$

- A $k \times k$ submatrix of **A**, formed by deleting $n - k$ columns, say $c_1, c_2, ..., c_{n-k}$, and the same $n - k$ rows, $r_1, r_2, ..., r_{n-k}$ from **A**, is called a "kth-order principal submatrix" of **A**
- The determinant of a $k \times k$ principal submatrix is called a "kth-order principal minor" of **A**
- The kth order principal submatrix of **A** formed by deleting the *last* $n - k$ columns and rows from **A** is called a "kth-order *leading* principal submatrix" of **A**
- The determinant of the kth-order leading principal submatrix of **A** is called the "kth order *leading* principal minor" (LPM) of **A**

Example: Principal minors of a matrix

Example: For a general 3×3 matrix $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

- There is one third-order principal minor: $|\mathbf{A}|$
- There are three second-order principal minors:

  (1) $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ formed by deleting column 3 and row 3 from $\mathbf{A}$

  (2) $\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$ formed by deleting column 2 and row 2 from $\mathbf{A}$

  (3) $\begin{vmatrix} a_{22} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$ formed by deleting column 1 and row 1 from $\mathbf{A}$

- There are three first-order principal minors:

  (1) $|a_{11}|$, formed by deleting the last 2 rows and columns

  (2) $|a_{22}|$, formed by deleting the first and third rows and columns

  (3) $|a_{33}|$, formed by deleting the first 2 rows and columns

Exercise: Find all the leading principal minors of $\mathbf{A}$ as functions of the elements of $\mathbf{A}$

Definiteness of symmetric matrices

We often care about the values taken by $\mathbf{x}'\mathbf{A}\mathbf{x}$ when $\mathbf{A}$ is a $n \times n$ symmetric matrix and $\mathbf{x} \neq \mathbf{0}$

Let $\mathbf{A}$ be a $n \times n$ symmetric matrix. Then:

(1) $\mathbf{A}$ is "positive definite" if $\mathbf{x}'\mathbf{A}\mathbf{x} > 0$ for all $\mathbf{x} \neq 0 \in \mathbb{R}^n$

- $\rightarrow$ iff all its n LPMs are strictly positive

(2) $\mathbf{A}$ is "positive semidefinite" if $\mathbf{x}'\mathbf{A}\mathbf{x} \geq 0$ for all $\mathbf{x} \neq 0 \in \mathbb{R}^n$

- $\rightarrow$ iff all its principal minors are non-negative

(3) $\mathbf{A}$ is "indefinite" if $\mathbf{x}'\mathbf{A}\mathbf{x} > 0$ for some $\mathbf{x} \in \mathbb{R}^n$, and < 0 for some other $\mathbf{x} \in \mathbb{R}^n$

(4) $\mathbf{A}$ is "negative semidefinite" if $\mathbf{x}'\mathbf{A}\mathbf{x} \leq 0$ for all $\mathbf{x} \neq 0 \in \mathbb{R}^n$

- $\rightarrow$ iff every principal minor of odd order is ≤ 0 and every principal minor of even order is ≥ 0
- $\rightarrow$ Alternatively, iff all principal minors of $-\mathbf{A}$ are non-negative

(5) $\mathbf{A}$ is "negative definite" if $\mathbf{x}'\mathbf{A}\mathbf{x} < 0$ for all $\mathbf{x} \neq 0 \in \mathbb{R}^n$

- $\rightarrow$ iff all its n LPMs alternate in sign as follows: $|\mathbf{A}_1| < 0$, $|\mathbf{A}_2| > 0$, $|\mathbf{A}_3| < 0$, etc.
- $\rightarrow$ Alternatively, iff all n LPMs of $-\mathbf{A}$ are strictly positive

Exercise: Definiteness of symmetric matrices

Exercise: Let $\mathbf{A} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$

(1) What are the principal minors of A?

(2) What is $\mathbf{x}'\mathbf{A}\mathbf{x}$ for a general $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$?

(3) What is the definiteness of **A**?

References

This drew notes from:

- Mathematics for Economists (Simon and Blume, 2015)